ISRAEL JOURNAL OF MATHEMATICS 80 (1992), 183–193

ON THE RESIDUALITY OF MIXING BY CONVOLUTIONS PROBABILITIES

BY

WOJCIECH BARTOSZEK*

Department of Mathematics and Applied Mathematics
Potchefstroom University for Christian Higher Education
2520 Potchefstroom, South Africa

ABSTRACT

A probability measure μ on a locally compact σ - compact amenable Hausdorff group G is called mixing by convolutions if for every pair of probabilities ν_1, ν_2 on G we have:

$$\lim_{n\to\infty} \|(\nu_1 - \nu_2) \star \mu^{\star n}\| = \lim_{n\to\infty} \|\mu^{\star n} \star (\nu_1 - \nu_2)\| = 0.$$

It is proved that the set of all mixing by convolutions probabilities is a norm (variation) dense subset of the set $P(G)$ of all probabilities on G. If G is additionally second countable the mixing measures are residual in $P(G)$.

1. Introduction

Let G be a locally compact σ–compact amenable Hausdorff group with a fixed left Haar measure λ. The Banach lattice (algebra with the convolution $\star$) of all real finite regular Borel measures on G is denoted by $M(G)$. For a measure $\nu \in M(G)$, $\|\nu\|$ is the total variation norm and $|\nu|$ is the modulus of ν. The convex, convolution semigroup of all regular probabilities on G is denoted by

---

* I would like to thank Professors A. Iwanik and R. Rębowski for sending preprints [I], [I–R], and [R] in which the residuality of $MIX_{\star L^1}$ was studied originally. I am also grateful to Professors J.J. Grobler and J.H. Fourie for participating in a seminar on which the first version of this paper was essentially improved.
Current address: Department of Mathematics, Applied Mathematics and Astronomy, University of South Africa, PO Box 392, 0001 Pretoria, South Africa.
Received May 29, 1991 and in revised form October 17, 1991

$P(G)$. Similarly, for every closed $K \subseteq G$, $P(K)$ is the set of all probabilities μ with $supp(\mu) \subseteq K$. The set of all probabilities with compact support is denoted by $P_c(G)$. As usual $C_b(G)$ denotes the Banach space of all bounded contionuous real valued functions on G with the supremum norm. We shall consider $P(G)$ as a topological space with respect to two topologies. The first is the one inherited from the Banach lattice $M(G)$ with the total variation norm. The second one is the weak topology i.e. the topology with base sets:

$$V_\mu(f_1,...,f_k,\varepsilon_1,...,\varepsilon_k) = \{\nu \in P(G) : |\int f_j d\nu - \int f_j d\mu| < \varepsilon_j, j = 1,...,k\}$$

where $f_1,...,f_k$ are from $C_b(G)$ and $\varepsilon_1,...,\varepsilon_k$ are positive numbers.

It is well known that if G is a polish group (metrizable separable and complete) then $P(G)$ with the weak topology is a polish space as well. In particular $P(G)$ is a Baire space then (see [P] for the details). We recall that every locally compact second countable Hausdorff group is a polish group (see Theorem 8.3 in [H–R 1]). By $\mathcal{L}(L^1(\lambda))$ we denote the Banach algebra of linear bounded operators on $L^1(\lambda)$ the Banach (convolution) algebra of all real finite signed measures absolutely continuous with respect to λ. An operator $T \in \mathcal{L}(L^1(\lambda))$ is called **stochastic** if: $T(f) \geq 0$ and $\|T(f)\| = \|f\|$ for all nonnegative $f \in L^1(\lambda)$. The set of all stochastic operators is denoted by $\mathcal{S}$. An important class of stochastic operators is the class of convolution operators. Recall, that for a probability measure μ on G, the operator $L^1(\lambda) \ni f \rightarrow T_\mu(f) = f \star \mu$ is called a **right convolution operator** (${}_\mu T(f) = \mu \star f$ is called a **left convolution operator**).

The importance of convolution operators is commonly recognized because of their coherence with the Markov processes on groups. Every *"time-space"* homogeneous Markov process (random walk) $\{\xi_n\}_{n\geq 0}$ is represented by some stochastic convolution operator. More precisely if $\{\xi_n\}_{n\geq 0}$ is such a Markov process, with transition probabilities $P(g, A) = \mu(g^{-1}A)$, then for every natural n and an initial distribution $f \in P(G) \cap L^1(\lambda)$ one has

$$P_f(\xi_n \in A) = \int_A T_\mu^n(f)\, d\lambda = \int_A T_{\mu^{\star n}}(f)\, d\lambda$$

where A is a Borel subset of G. In this paper some concepts of asymptotic behaviour of random walks on amenable groups are investigated. We consider the iterates $T_\mu^n(f)$ or ${}_\mu T^n(f)$ and study their dependence on the initial density f. The set of measures $\mu \in P(G)$ for which the distributions $T_\mu^n(f_1)$, $T_\mu^n(f_2)$ are asymptotically close independently of the starting ones f_1, f_2 is the main subject of our considerations. There are three operator topologies in $\mathcal{L}(L^1(\lambda))$ which are helpful in this task: the operator norm topology (o.n.t.), the strong operator topology (s.o.t.), and the weak operator topology (w.o.t.). Using Wendel's Theorem (see [H–R 2]) it was noticed in [I–R] that the class of right (or left) convolution operators is s.o.t. Baire. Let $M_{L^1\star}$ (respectively $M_{\star L^1}$) denote the set of all right convolution operators on $L^1(\lambda)$ (left convolution operators on $L^1(\lambda)$). According to [I–R] and [R] the operator $T_\mu \in M_{L^1\star}$ (or ${}_\mu T \in M_{\star L^1}$) is called **norm completely mixing** if for each pair of probabilities $\nu_1, \nu_2 \in L^1(\lambda)$

$$(1_r) \qquad \lim_{n\to\infty} \|T_\mu^n(\nu_1-\nu_2)\| = \lim_{n\to\infty} \|(\nu_1-\nu_2)\star\mu^{\star n}\| = 0$$

$$(\text{or } (1_l) \qquad \lim_{n\to\infty} \|{}_\mu T^n(\nu_1-\nu_2)\| = \lim_{n\to\infty} \|\mu^{\star n}\star(\nu_1-\nu_2)\| = 0).$$

The set of all norm completely mixing right (or left) convolution operators on $L^1(\lambda)$ is denoted by $MIX_{L^1\star}$ (or $MIX_{\star L^1}$ repectively). If $T_\mu \in MIX_{L^1\star}$ (or ${}_\mu T \in MIX_{\star L^1}$) we simply say that the probability μ is **right (left)** L^1–**mixing by convolutions** and denote the set of all such measures by $mix_{L^1\star}$ (or $mix_{\star L^1}$). A probability measure μ on G is called L^1- **mixing by convolutions** if $\mu \in mix_{L^1\star} \cap mix_{\star L^1} = mix(L^1)$. It was proved in [RO] that there exists a right L^1–mixing by convolutions probability on a locally compact, Hausdorff group G if and only if G is σ-compact and amenable. If G is additionally abelian and second countable, it was recently observed that $MIX_{\star L^1}$ is a dense G_δ in $M_{\star L^1}$ in both the strong operator and the operator norm topologies (see Theorems 3 and 5 in [I–R]). It is our aim to extend this result of Iwanik and Rębowski to all amenable locally compact polish groups.

In the first part of our paper a stronger version of mixing is considered. Namely a measure $\mu \in P(G)$ is called **right (left) mixing by convolutions** if for each pair of probabilities $\nu_1, \nu_2 \in P(G)$ one has

$$(2_r) \quad \lim_{n\to\infty} \|(\nu_1-\nu_2)\star\mu^{\star n}\| = 0 \quad (\text{ or } \quad (2_l) \quad \lim_{n\to\infty} \|\mu^{\star n}\star(\nu_1-\nu_2)\| = 0 \ ).$$

The set of all probabilities on G satisfying (2_r) (or (2_l) respectively) is denoted by $mix_{M\star}$ (or $mix_{\star M}$). The intersection $mix_{M\star} \cap mix_{\star M}$ is denoted by $mix(M)$ and measures from the last set are called **mixing by convolutions**. Obviously the following inclusions hold: $mix_{M\star} \subseteq mix_{L^1\star}$, $mix_{\star M} \subseteq mix_{\star L^1}$ and $mix(M) \subseteq mix(L^1)$. Let us notice that these inclusions are proper even for G to be the one dimensional torus. Indeed, the measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{exp(\frac{2\pi i}{n})}$ is L^1- mixing by convolutions (see [L]) but for every irrational $\omega \in \mathbb{R}$ and arbitrary natural n we have $\|(\delta_{exp(2\pi i\omega)} - \delta_1)\star\mu^{\star n}\| = 2$, so μ is not mixing by convolutions.

We show that for every locally compact σ− compact Hausdorff and amenable group G the set $mix(M)$ is large. Namely, by our Theorem 1, the norm variation closure of $mix(M)$ is $P(G)$. Applying this fact to the convolution operators we obtain that $MIX_{L^1\star}$ is norm operator dense in $M_{L^1\star}$. Moreover if G is additionally second countable then $MIX_{L^1\star}$ is a G_δ in the w.o.t. (so in the s.o.t. and the o.n.t. as well).

Finally let us remark that $P(G)$ with the variation norm topology and $M_{L^1\star}$ with the operator norm topology are homeomorphic (by the existence of an approximative unit in $L^1(\lambda)$ they are isometric even). $P(G)$ with the weak measure topology is homeomorphic with $M_{L^1\star}$ equipped with the weak or strong operator toplogy (for this fact we can use the Theorem 1.1.9 and the Lemma 2.5.13 from [H]).

CONVENTION: *All topological groups considered in this paper are at least locally compact σ-compact Hausdorff and amenable. The measures are Borel and regular.*

2. Existence and denseness of mixing measures

The following idea of mixing will be helpful in the sequel. Namely let $\alpha_n \to 0$ be a sequence of positive numbers. Then,

DEFINITION 1: *A probability measure μ on G is called right (left) mixing by convolutions with the rate α_n if for every compact set $K \subseteq G$ there exists a natural number N_K such that for each pair of probabilities $\nu_1, \nu_2 \in P(K)$ and $n \geq N_K$ the following inequality*

$$(3_r) \quad \|(\nu_1 - \nu_2)\star\mu^{\star n}\| \leq \alpha_n \quad (\textit{or} \quad (3_l) \quad \|\mu^{\star n}\star(\nu_1 - \nu_2)\| \leq \alpha_n) \quad \textit{holds.}$$

The set of all such probabilities is denoted by $mix_{M\star,\alpha_n}$ (or by $mix_{\star M,\alpha_n}$ respectively).

Remark 1: Since $P_c(G)$ is a norm variation dense subset of $P(G)$ and the operators T_μ are contractions on $M(G)$ we have $mix_{\star M,\alpha_n} \subseteq mix_{\star M}$, $mix_{M\star,\alpha_n} \subseteq$

$mix_{M\star}$ and $mix(M)_{\alpha_n} \subseteq mix(M)$, where $mix(M)_{\alpha_n} = mix_{\star M,\alpha_n} \cap mix_{M\star,\alpha_n}$. Let us notice that $\sim: M(G) \to M(G)$ defined as $\tilde{\mu}(A) = \mu(A^{-1})$ is a linear isometry onto so $(mix_{L^1\star})^{\sim} = mix_{\star L^1}$ and $(mix_{M\star})^{\sim} = mix_{\star M}$. The mapping $\sim$ is a homeomorphism of $P(G)$ in the weak measure topology as well. Hence it is justifiable to consider only the right mixing . ∎

Recall (see [E–G] and [E]) that the amenability of G is equivalent to the following Emerson condition: for every $\varepsilon > 0$ and every compact set $K \subseteq G$ there exists a compact symmetric set $S \subseteq G$ with $0 < \lambda(S) < +\infty$ such that for all $g \in K$ we have $\lambda(gS\Delta S) \leq \varepsilon\lambda(S)$ where Δ denotes the symmetric difference.

The following Lemma 1 reformulates the above condition somewhat.

LEMMA 1: *Let K be a compact subset of the group G. For every $\varepsilon > 0$ there exists a symmetric, compact set $S_{K,\varepsilon}$ such that if $\nu_1, \nu_2 \in P(K)$ then*

$$\|(\nu_1 - \nu_2) \star \chi_{S_{K,\varepsilon}}\| \leq \varepsilon\lambda(S_{K,\varepsilon}). \tag{4}$$

Proof: Let $S_{K,\varepsilon}$ be a symmetric, compact set from the Emerson's characterization with $\frac{\varepsilon}{2}$ instead of ε and $\chi_{S_{K,\varepsilon}}$ be its characteristic function. Then

$$\begin{aligned}
\|(\nu_1 - \nu_2) \star \chi_{S_{K,\varepsilon}}\| &= \int_G \left| \int_G \chi_{S_{K,\varepsilon}}(y^{-1}x)\, d(\nu_1 - \nu_2)(y) \right| d\lambda(x) \\
&= \int_G \left| \int_G (\chi_{S_{K,\varepsilon}}(y^{-1}x) - \chi_{S_{K,\varepsilon}}(x))\, d(\nu_1 - \nu_2)(y) \right| d\lambda(x) \\
&\leq \int_G \int_G |\chi_{yS_{K,\varepsilon}}(x) - \chi_{S_{K,\varepsilon}}(x)|\, d|\nu_1 - \nu_2|(y)\, d\lambda(x) \\
&= \int_K \lambda(yS_{K,\varepsilon}\Delta S_{K,\varepsilon})\, d|\nu_1 - \nu_2|(y) \leq \varepsilon\lambda(S_{K,\varepsilon}). \ \blacksquare
\end{aligned}$$

For a compact set $K \subseteq G$ and positive ε let $R_{K,\varepsilon}$ ($L_{K,\varepsilon}$) denote the set of all probabilities $\mu \in L^1(\lambda)$ such that (4_r) $\|(\nu_1 - \nu_2)\star\mu\| \leq \varepsilon$ (or$(4_l)\|\mu\star(\nu_1 - \nu_2)\| \leq \varepsilon$ respectively) for all $\nu_1, \nu_2 \in P(K)$.

By Lemma 1 the set $R_{K,\varepsilon}$ is nonempty. Clearly it is closed in the L^1 norm and convex. Notice that $L_{K,\varepsilon} = (R_{K^{-1},\varepsilon})^{\sim}$, so the set $L_{K,\varepsilon}$ has same property. If K is compact and symmetric then $\mu \in R_{K,\varepsilon}$ if and only if $\tilde{\mu} \in L_{K,\varepsilon}$.

Notice that $R_{K,\varepsilon} \star P(G) \subseteq R_{K,\varepsilon}$ and $P(G) \star L_{K,\varepsilon} \subseteq L_{K,\varepsilon}$ so the set (5) $B_{K,\varepsilon} = R_{K,\varepsilon} \cap L_{K,\varepsilon}$ is nonempty (it contains $R_{K,\varepsilon} \star L_{K,\varepsilon}$). The following three inclusions will be useful in the sequel: $(6_r) R_{K_2,\varepsilon_2} \subseteq R_{K_1,\varepsilon_1}$, $(6_l) L_{K_2,\varepsilon_2} \subseteq L_{K_1,\varepsilon_1}$ and (6) $B_{K_2,\varepsilon_2} \subseteq B_{K_1,\varepsilon_1}$ for $K_2 \supseteq K_1$ and $\varepsilon_2 \leq \varepsilon_1$.

Now we introduce a class $\mathcal{A}$ of positive sequences (α_n) converging to 0 with the following property: there exists a decreasing to 0 sequence $0 < r_n \le 1$ such that for every $0 < \varepsilon \le 1$

$$\lim_{n\to\infty} \frac{(1 - \varepsilon r_{n-1})^n}{\alpha_n} = 0 . \tag{7}$$

It is rather an elementary fact that the class $\mathcal{A}$ coincides with

$$\{ (\alpha_n) : \exists_{0<a<1} \exists_{\lambda_n\to 0} \text{ with } n\lambda_n \to \infty \text{ such that } \alpha_n = a^{n\lambda_n}\}.$$

The following Theorem 1, which is the main result of our paper is a generalization of the Theorem 1.10 from [R]. The first phrases of our proof can be recognized as some pieces of Rosenblatt's proof. However for the reader's convenience and the completeness of the paper a full proof is given here. Moreover, our approach to this problem seems to be more natural and effective than the one presented in [RO]. We remark that, in the abelian case, if a measure μ_0 is mixing by convolutions then for every $0 < \varepsilon \le 1$ and probability measure μ on G the convex combination $(1-\varepsilon)\mu + \varepsilon\mu_0$ is mixing by convolutions (for L^1-mixing we may apply results from [F] or [L]). In the following Theorem 1 only the amenability and σ-compactness of G are assumed to obtain a similar result. Namely we prove that for some measure μ_0 on G any convex combination $(1-\varepsilon)\mu+\varepsilon\mu_0$ belongs to $mix(M)_{\alpha_n}$ if μ is a compactly supported probability on G, $0 < \varepsilon \le 1$ and $(\alpha_n) \in \mathcal{A}$. In particular the norm denseness of $mix(M)_{\alpha_n}$ in $P(G)$ is easily seen.

THEOREM 1: *For every $(\alpha_n) \in \mathcal{A}$ there exists an absolutely continuous, symmetric measure μ_0 (i.e. $\tilde{\mu}_0 = \mu_0$) such that for every $0 < \varepsilon \le 1$ and $\mu \in P_c(G)$ the measure $(1-\varepsilon)\mu + \varepsilon\mu_0$ belongs to $mix(M)_{\alpha_n}$.*

Proof: By the σ-compactness of G we may choose an increasing sequence of symmetric compact sets $D_n \subseteq G$ such that $\bigcup_{n=1}^{\infty} Int(D_n) = G$. Assume the neutral element of G belongs to D_1. Let (γ_n) be a decreasing to 0 sequence of positive numbers such that $0 < \gamma_n \le 2^{-n}\alpha_n$. We begin by constructing (inductively) two sequences of compact symmetric sets K_n, $S_n \subseteq G$. We set $K_1 = D_1$ and $S_1 = S_{K_1,\gamma_1}$. If $K_1, K_2, \ldots, K_{n-1}$ and $S_1, S_2, \ldots, S_{n-1}$ are given we define

$$K_n = K_{n-1} \cup D_n(D_n \cup S_1S_1 \cup \cdots \cup S_{n-1}S_{n-1})^{n-1}$$
$$\cup (D_n \cup S_1S_1 \cup \cdots \cup S_{n-1}S_{n-1})^{n-1}D_n$$

and $S_n = S_{K_n,\gamma_n}$. Now let $r_0 = 1$ and $r_n \searrow 0$ be such that (7) holds. The sets K_n are symmetric and so by Lemma 1 and the properties gathered in (5) and (6) we have: for all $n \geq 1$

$$\frac{\chi_{S_n}}{\lambda(S_n)} \star \left(\widetilde{\frac{\chi_{S_n}}{\lambda(S_n)}}\right) = \frac{d\,\mu_n}{d\,\lambda} \in B_{K_n,\gamma_n}.$$

We show that the measure $\mu_0 = \sum_{n=1}^{\infty} (r_{n-1} - r_n)\mu_n$ satisfies the property described in the statement of our Theorem 1. Let μ , ν_1 , $\nu_2 \in P_c(G)$ be arbitrary and N be such that for all $n \geq N$ the inclusion $supp\,(\mu+\nu_1+\nu_2) \subseteq D_n$ holds. For a fixed $0 < \varepsilon \leq 1$ we introduce the following notations: $\rho_{n,0} = (1-\varepsilon)\mu + \varepsilon\sum_{j=1}^{n-1}(r_{j-1} - r_j)\mu_j$ and $\rho_{n,1} = \varepsilon\sum_{j=n}^{\infty}(r_{j-1} - r_j)\mu_j$. Clearly $\|\rho_{n,0}\| = 1 - \varepsilon r_{n-1}$, $\|\rho_{n,1}\| = \varepsilon r_{n-1}$ and $\frac{\rho_{n,1}}{\varepsilon r_{n-1}} \in R_{K_n,\gamma_n}$ (the last easily follows from the properties (5) and (6) and the fact that the sequence γ_n is decreasing and K_n is increasing). Now let us start to estimate:

$$\begin{aligned}
\varepsilon_n &= \|\,(\nu_1 - \nu_2) \star ((1-\varepsilon)\mu + \varepsilon\mu_0)^{\star n}\| \\
&= \|\,(\nu_1 - \nu_2) \star (\rho_{n,0} + \rho_{n,1})^{\star n}\,\| \\
&\leq \|\,(\nu_1 - \nu_2) \star \rho_{n,0}^{\star n}\,\| + \|\,(\nu_1 - \nu_2) \star \sum_{j=1}^{n} \sum_{q_1+q_2+\ldots+q_n=j} \rho_{n,q_1} \star \ldots \star \rho_{n,q_n}\,\| \\
&\leq 2\|\,\rho_{n,0}\,\|^n + \sum_{j=1}^{n} \sum_{q_1+q_2+\ldots+q_n=j} \|\,(\nu_1 - \nu_2) \star \rho_{n,q_1} \star \ldots \star \rho_{n,q_n}\,\| \,.
\end{aligned}$$

The first term in the last inequality is exactly $2(1-\varepsilon r_{n-1})^n$ and the second term can be estimated by

$$\begin{aligned}
&2^n\, sup_{0\leq j<n}\, \|\,(\nu_1 - \nu_2) \star \rho_{n,0}^{\star j} \star \rho_{n,1}\| \\
&\leq 2^n\, sup_{\tau_1,\tau_2\in P(K_n)}\, \varepsilon r_{n-1}\, \|(\tau_1 - \tau_2) \star \frac{\rho_{n,1}}{\varepsilon r_{n-1}}\,\| \leq 2^n r_{n-1}\frac{\alpha_n}{2^n} = r_{n-1}\alpha_n
\end{aligned}$$

(notice that $\nu_k \star (\frac{\rho_{n,0}}{1-\varepsilon r_{n-1}})^{\star j}$ are supported on K_n , here $k = 1, 2$, $0 \leq j < n$ and $n > N$). Finally, for N large enough, we have for all $n > N$ $2(1-\varepsilon r_{n-1})^n < \frac{1}{2}\alpha_n$, $r_{n-1} < \frac{1}{2}$ and $\varepsilon_n \leq \alpha_n$. Consequently, $(1-\varepsilon)\mu + \varepsilon\mu_0 \in mix_{M\star,\alpha_n}$. It can be shown analogously that $(1-\varepsilon)\mu + \mu_0 \in mix_{\star M,\alpha_n}$ so the proof of the Theorem 1 is completed. ∎

Remark 2: We notice that if the measure μ (in Theorem 1) is taken to be absolutely continuous , then the convex combination $(1-\varepsilon)\mu + \varepsilon\mu_0$ is again absolutely continuous. In particular the set $mix(M) \cap L^1(\lambda)$ is norm dense in $P(G) \cap L^1(\lambda)$ as well. Now the following Corollary 1 is a simply consequence of our Theorem 1. ∎

COROLLARY 1: *For any* $(\alpha_n) \in \mathcal{A}$ *we have :* $\overline{mix(M)_{\alpha_n}}^{\|\ \|} = P(G)$ *and* $\overline{mix(M)_{\alpha_n} \cap L^1(\lambda)}^{\|\ \|} = P(G) \cap L^1(\lambda)$.

Remark 3: The rate of convergence of $\|(\nu_1 - \nu_2) \star \mu^{\star n}\| \to 0$ which can be obtained using our Theorem 1 is not exponential, but it seems to be fast enough from the probabilistic point of view. For instance if $\alpha_n = a^{n\lambda_n}$ where $0 < a < 1$ and $\frac{n\lambda_n}{ln(n)} \to +\infty$ then for every $\mu \in mix(M)_{\alpha_n}$ and a compact subset $K \subseteq G$ we have

$$\sum_{n=1}^{\infty} sup_{\nu_1,\nu_2 \in P(K)} \ n^k \|(\nu_1 - \nu_2) \star \mu^{\star n}\| < \infty.$$

In order to check it we notice that $\sum_{n=1}^{\infty} n^k \alpha_n < \infty$. ∎

3. Residuality of mixing measures

Assume that G is an amenable locally compact polish group. Obviously we have the following representation of right L^1- mixing by convolutions measures:

$$(8_r) \qquad mix_{L^1\star} = \bigcap_{l,k} \bigcap_{m} \bigcap_{N} \bigcup_{n \geq N} \{\mu \in P(G) : \|(\nu_l - \nu_k) \star \mu^{\star n}\| < \frac{1}{m}\}$$

where $\{\nu_1, \nu_2, \dots\}$ is an L^1 norm dense subset of $P(G) \cap L^1(\lambda)$. From this it is easily seen that $mix_{L^1\star}$ is a weak G_δ in $P(G)$. A similar representation (8_l) for $mix_{\star L^1}$ shows that the set of left L^1-mixing by convolutions measures is also a weak G_δ. In particular $mix(L^1)$, as the intersection of two G_δ- sets is a weak G_δ. Now we are in position to formulate the following category result.

THEOREM 2: *For every locally compact, amenable and polish group G we have: (9) $mix(L^1)$ is a dense G_δ in $P(G)$ for both the weak and the variation norm topologies on $P(G)$ and (10) $mix(L^1) \cap L^1(\lambda)$ is a dense G_δ in $P(G) \cap L^1(\lambda)$ for the L^1-norm topology.*

Proof: It was noticed in the Corollary 1 that $mix(L^1)$ and $mix(L^1) \cap L^1(\lambda)$ are dense subsets for these topologies. The proof that they are G_δ-sets was presented just before the formulation of this Theorem. ∎

Remark 4: We do not consider the residuality of $mix(L^1) \cap L^1(\lambda)$ in the weak topology since $P(G) \cap L^1(\lambda)$ can be meager in itself for this topology. In fact, assume that $U_n \subseteq G$ is a decreasing sequence of dense and open subsets of G with $\lambda(U_n) \searrow 0$ and let $F_n = \{\varrho \in P(G) \cap L^1(\lambda) : \varrho(G \setminus U_n) \geq \frac{1}{2}\}$. Clearly every set F_n is closed in the weak topology on $P(G) \cap L^1(\lambda)$. Since U_n is dense and open it follows from Theorem 6.3 in [P] that the absolutely continuous measures with supports in U_n are weakly dense in $P(G)$. This means that $P(G) \cap L^1(\lambda) = \bigcup_{n=1}^{\infty} F_n$ is a space of the first category. The next Corollary 2 elucidate this case quite thoroughly. ∎

COROLLARY 2 : *Let G be an amenable locally compact polish group and denote by $P_s(G)$ the set of all singular (with respect to the Haar measure) probabilities on G. If the topology on G is not discrete then $mix(L^1) \cap P_s(G)$ contains a weak dense G_δ.*

Proof: It is sufficient to notice that $P_s(G)$ contains a weak dense G_δ. As in the Remark 4 we choose a decreasing sequence of dense and open sets $U_n \subset G$ with $\lambda(U_n) \searrow 0$. The set of all probabilities on G with nonzero absolutely continuous component is included in the countable union $\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} F_{k,n}$ where

$$F_{k,n} = \{ \varrho \in P(G) : \varrho(G \setminus U_n) \geq \frac{1}{k} \}$$

are weakly closed and nowhere dense. Obviously the following inclusion

$$\bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} (P(G) \setminus F_{k,n}) \subseteq P_s(G)$$

holds. Since $P(G)$ with the weak topology is a polish space $mix(L^1) \cap P_s(G)$ contains a dense weak G_δ of the form $mix(L^1) \cap \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} (P(G) \setminus F_{k,n})$. ∎

The results obtained in this note are usefull in creating L^1–mixing by convolutions measures with some additional properties. The following Corollaries are good examples of this. Let us recall (see [P] , Definition 4.1) that a probability measure μ on a group G is indecomposable if there do not exist two nondegenerate (not δ_g where $g \in G$) probabilities μ_1 , μ_2 with $\mu = \mu_1 \star \mu_2$. Assume that G is an infinite polish group. Then the set of all indecomposable probabilities on G (denoted by $P_I(G)$) is a dense G_δ in $P(G)$ in the weak topology. If in addition G is uncountable then a weak dense G_δ is the set $P_{I,1}$ of all nonatomic and indecomposable probabilities (see [P], Theorems 4.3 and 4.4).

COROLLARY 3: *Let G be an infinite locally compact polish amenable group. Then : (11) $mix(L^1) \cap P_I(G)$ is a weak dense G_δ in $P(G)$ and (12) if G is in addition uncountable then $mix(L^1) \cap P_{I,1}(G)$ is a weak dense G_δ in $P(G)$.*

The next result is a simple application of (10) and Theorem 12.1 from [P].

COROLLARY 4: *For every locally compact, noncompact abelian polish group G the set $P_I(G) \cap mix(L^1) \cap L^1(\lambda)$ is a dense G_δ in $P(G) \cap L^1(\lambda)$ for the L^1-norm topology.*

We finish our consideration with the following Theorem 3 which provides an affirmative answer to a question raised by A.Iwanik.

THEOREM 3: *Let G be a locally compact, second countable, Hausdorff amenable group. Then $MIX_{L^1\star}$ (and $MIX_{\star L^1}$) is a dense G_δ in $M_{L^1\star}$ ($M_{\star L^1}$ respectively) in the operator norm, the strong operator and the weak operator topologies.*

Proof: By Theorem 8.3 of [H–R 1] G is completely metrizable and separable, and therefore polish. Since $M_{L^1\star}$ is homeomorphic to $P(G)$ with respect to the appropriate topologies, an application of Theorem 2 yields the desired result. ∎

We end our paper with the following three Remarks.

Remark 5: If G is not second countable we may not represent $mix_{L^1\star}$ and $mix_{M\star}$ as in (8_r). However for the norm variation topology these sets are still dense G_δ- sets. It follows from the following representations:

$$mix_{M\star} = \bigcap_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \{\mu : \ sup_{\nu_1,\nu_2 \in P(K_l)} \|(\nu_1 - \nu_2) \star \mu^{\star n}\| < \frac{1}{m}\}$$

and

$$mix_{L^1\star} = \bigcap_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \{\mu : \ sup_{\nu_1,\nu_2 \in P(K_l) \cap L^1(\lambda)} \|(\nu_1 - \nu_2) \star \mu^{\star n}\| < \frac{1}{m}\}.$$

∎

Remark 6: It is noticed in [RO] (see p.37) that for a σ-compact locally compact amenable and unimodular Hausdorff group G there exists $f \in L^1(\lambda) \cap P(G)$ such that for all $h_1, h_2 \in L^1(\lambda) \cap P(G)$ we have

$$\lim_{n \to \infty} \|(h_1 - h_2) \star f^{\star n}\| = \lim_{n \to \infty} \|f^{\star n} \star (h_1 - h_2)\| = 0.$$

Our Theorem 1 shows that the unimodularity assumption was not essential. Moreover Rosenblatt's existence result is now replaced by the denseness of such measures (and if G is in addition second countable, by our Theorem 2 such measures form a norm dense G_δ subset of $L^1(\lambda) \cap P(G)$ even). ∎

Remark 7: Recently, the author has been informed that similar result to our Theorem 1 was obtained by R. Rębowski. It is proved in [R] that for any second countable, locally compact, and nilpotent group G if $\mu \in mix_{L^1 *}$ is spread out then for any positive $0 < \varepsilon \leq 1$ and any $\nu \in P(G)$ the convex combination $\mu_\varepsilon = \varepsilon\mu + (1-\varepsilon)\nu$ is right L^1-mixing by convolutions. ∎

References

[E] W.Emerson, *Large symmetric sets in amenable groups and the individual ergodic theorem,* Am. J. Math. **96** (1974), 242–247.

[E–G] W.Emerson and F.Greenleaf, *Covering properties and Følner Conditions for locally compact groups,* Math. Z. **102** (1967), 370–384.

[F] S.R.Foguel, *On iterates of convolutions,* Proc. Am. Math. Soc. **47** (1975), 368–370.

[H] H.Heyer, *Probability measures on Locally Compact Groups,* Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[H–R 1] E.Hewitt and K.Ross, *Abstract Harmonic Analysis,* Vol.1, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1963.

[H–R 2] E.Hewitt and K.Ross, *Abstract Harmonic Analysis,* Vol.2, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[I] A.Iwanik, *Baire category of mixing for stochastic operators,* Rend. Circ. Mat. Palermo II Ser. Proceedings of the Conference on Measure Theory, Oberwolfach (1990).

[I–R] A.Iwanik and R.Rębowski, *Structure of mixing and category of complete mixing for stochastic operators,* Annales Pol. Math. **56** (1992), 233–242.

[L] M.Lin, *Convergence of convolution powers on a LCA group,* Semestrbericht Funktionanalysis Wintersemestr 82/83 Tübingen.

[P] K.Parthasarathy, *Probability Measures on Metric Spaces,* Academic Press, New York and London, 1967.

[R] R. Rębowski, *Most random walks on nilpotent groups are mixing,* Annales Pol. Math., to appear.

[RO] J.Rosenblatt, *Ergodic and mixing random walks on locally compact groups,* Math. Ann. **257** (1981), 31–42.